Exploiting duality in a toy model of QCD at non-zero temperature and chemical potential: the massive Thirring model, sine-Gordon model and Coulomb gases.

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Abstract

We focus on the massive Thirring model in 1+1 dimensions at finite temperature T and non-zero chemical potential μ, and comment on some parallels between this model and QCD. In QCD, calculations of physical quantities such as transport coefficients are extremely difficult. In the massive Thirring model, similar calculations are greatly simplified by exploiting the duality which exists with the sine-Gordon model and its relation, at high T, to the exactly solvable classical Coulomb gas on the line.

1 Introduction and motivation

The massive Thirring (MT) model in 2D Euclidean space with metric $(+,+)$ is our toy model of QCD. It is described by the fermionic Lagrangian

$$\mathcal{L}_{MT}[\bar{\psi},\psi] = i\bar{\psi}(\partial\!\!\!/ - m_0)\psi + \frac{1}{2}g^2 j_\mu(x)j^\mu(x) + \mu j_0(x), \qquad (1)$$

where $j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ is the conserved current and μ the chemical potential. As in QCD, (1) is only invariant under chiral transformations $\psi \to e^{ia\gamma_5}\psi$ if $m_0 = 0$, and later we will study chiral symmetry restoration in this model as $T \to \infty$, as a function of μ and coupling constant g^2. We always take $g^2 > 0$ so that the attractive interaction gives fermion anti-fermion bound states which correspond to hadrons in QCD.

The sine-Gordon (SG) model, on the other hand, with Lagrangian

$$\mathcal{L}_{SG}[\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\alpha_0}{\lambda^2}\cos\lambda\phi, \tag{2}$$

will play the part of the effective chiral bosonic lagrangian for low energy QCD. There are an infinite number of degenerate vacua $\phi_v = 2n\pi/\lambda$ ($n \in \mathbb{Z}$) and hence kink solutions. These correspond to skyrmions (baryons) in low energy QCD. Note that the Lagrangian (2) is invariant under $\phi \to \phi + \phi_v$ (the counterpart of isospin symmetry in QCD), whilst the potential term breaks explicitly the symmetry $\phi \to \phi + a$, which is the counterpart of the chiral symmetry in the MT model.

At $T = \mu = 0$ the SG and MT models are equivalent [1, 2] and dual provided

$$\frac{\lambda^2}{4\pi} = \frac{1}{1 + g^2/\pi}. \tag{3}$$

Thus perturbative calculations in one theory tell us about non-perturbative effects in the other. Further, the weak identity

$$j_\mu(x) \leftrightarrow \frac{\lambda^2}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi(x) \tag{4}$$

shows that fermions in the MT model correspond to kinks in the SG model (cf baryons and skyrmions in QCD). The bound states in the MT model correspond to bosons and kink anti-kink breather solutions in the SG model.

How are these links affected when $T > 0$ and $\mu \neq 0$? One can show[3, 4] that at $T > 0$, $\mu = 0$ the partition functions of the two models are the same, $Z_{SG}(T, \mu = 0) = Z_{MT}(T, \mu = 0)$. When $\mu \neq 0$, the equivalence $Z_{SG}(T, \mu) = Z_{MT}(T, \mu)$ holds[4], where $Z_{SG}(T, \mu)$ is now the partition function for the SG model with a topological term which counts the number of kinks minus anti-kinks; $\mathcal{L}_{SG}(\mu) = \mathcal{L}_{SG} - \mu\frac{\lambda}{2\pi}\frac{\partial\phi}{\partial x}$.

2 Massive Thirring model as a Coulomb Gas

At any T the MT model is not only equivalent to the SG model, but also to non-relativistic particles of charge $\pm q$ (a neutral Coulomb gas (CG)) on a cylinder, circumference T^{-1}. When the cylinder collapses to a line at temperatures $T \gtrsim m$, the equivalent[5] 1D neutral CG can be solved exactly.

To see the equivalence, recall some of the properties of a 1D CG at a temperature T. The potential which binds the charges q_i ($=\pm q$) at positions x_i is $V(x_i, x_j) = -2\pi q_i q_j |x_i - x_j|$, so the grand canonical partition function for the system confined on a line of length L is

$$\Omega(z, T, q, L) = \sum_{N=0}^{\infty} \frac{z^{2N}}{(N!)^2} \left(\prod_{i=1}^{2N} \int_0^L dq_i \right) \exp\left[2\pi q^2 T^{-1} \sum_{1 \leq j < i \leq 2N} \epsilon_i \epsilon_j |q_i - q_j| \right]. \tag{5}$$

Here $\epsilon_i = 1$ for $i \leq N$ and $\epsilon_i = -1$ for $i > N$. This classical problem can be solved analytically[6, 7]. As a result one finds that, at low T, the CG behaves as a free gas of 'molecules' made up of $+-$ charge pairs bound together whereas, at high T, the charges are deconfined and the pressure is large.

How is the MT model partition function related to (5)? A hint comes from noting that in perturbation theory about $\alpha_0 = 0$, $Z_{SG}(T)$ is a sum of terms each of which contains the expectation value of products of $\cos \lambda\phi$'s. Written in exponential form, these are just products of free massless boson propagators which behave[5] as $\Delta(T) \sim T|x|$ for $T \gtrsim m$. Thus one indeed gets a series of the form (5). More exactly,

$$Z_{MT}(T, L) = Z_0^F(T, L)\Omega(z = f(m, T, g^2), T, q \propto T, L), \tag{6}$$

where $Z_0^F(T, L)$ is the partition function for free massless fermions and f is a specified function[5] whose form is unimportant here. The important point to note from (6) is that $q \propto T$. This implies that the CG behaviour above is reversed. Furthermore one can show[5] that the CG charges correspond in the MT model to 'chiral' charges $\pm q \leftrightarrow \hat{\sigma}_\pm = \bar{\hat{\psi}}(1 \pm \gamma_5)\hat{\psi}$. Hence we are lead to an interesting picture of the chiral symmetry properties of the MT model: at *low* T the system is in a 'plasma phase' of free $\hat{\sigma}_\pm$ charges and chiral symmetry is broken. At *high* T the system is in the 'molecular phase' in which the $\hat{\sigma}_\pm$ bind into chirally invariant 'molecules' $\hat{\sigma}_+\hat{\sigma}_-$ so that chiral symmetry is restored.

This understanding can be verified by calculating the chiral condensate $\ll \bar{\psi}\psi \gg$. The result is plotted in Fig.1 for different values of g^2. Note that if $g^2 = 0$ chiral symmetry can never be restored since there is not sufficient energy to bind the molecules. For larger g^2 chiral symmetry is restored asymptotically as $T \to \infty$. Similarly one can calculate the pressure exactly in this $T \gtrsim m$ limit: it is given by $P_{MT} = \pi T^2/6 + P_{CG}(g^2, T)$ where the first term is the contribution from the free massless fermions in (6), and the CG part is given by

$$P_{CG} = \frac{2\pi T^2}{1 + g^2/\pi}\gamma_0 \left[\frac{m^2}{4\pi T^2} \left(1 + \frac{g^2}{\pi}\right) \left(\frac{2T}{m}\right)^{\frac{1}{1+g^2/\pi}} \right], \tag{7}$$

where γ_0 is the highest eigenvalue of the Mathieu differential equation[5]. We stress that this result is non-perturbative and exact for $T \gtrsim m$ — it would be interesting to compare the result with an order by order calculation in perturbation theory. Similar comments apply for $\mu \neq 0$ when the CG picks up a contribution from an (imaginary) external electric field[5]; one can then, for example, obtain the net averaged fermion density $\rho(T, \mu)$ exactly.

3 Transport coefficients in the MT model

Finally one can try to calculate transport coefficients in the MT model, exploiting as much as possible the duality with the SG model and the link with a 1D CG. As an example of the power of this approach, consider the response of the MT model to an external classical electro-magnetic potential A^μ_{Cl}. From linear response theory

$$\ll \delta j_\mu(x, t) \gg = -i \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' A_0^{cl}(x', t') \ll [j_\mu(x, t), j_0(x', t')] \gg \theta(t - t'), \tag{8}$$

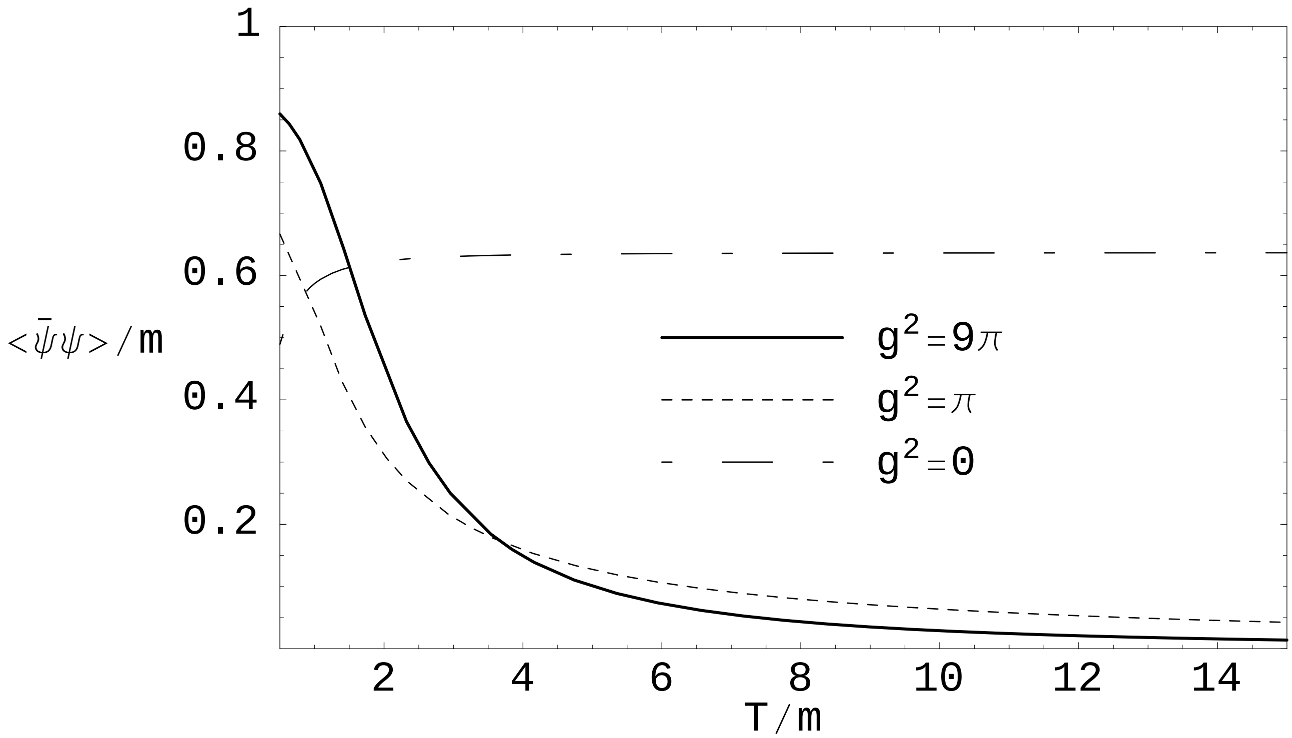

Figure 1: The Fermion condensate as a function of T for different g^2.

which, on using the duality with the SG model gives the conductivity (in momentum space)[8]

$$\delta j_1(k_0,k)=\sigma(k_0,k)E^{Cl}(k_0,k)=\frac{1}{1+g^2/\pi}k_0 D_R^{SG}(k_0,k)E^{Cl}(k_0,k) \tag{9}$$

Here $D_R^{SG}(k_0,k)$ is the retarded *bosonic* propagator of the SG model: that is, we have rewritten the fermionic current-current correlator in terms of bosonic collective modes (a quasi-particle propagator). The existence of such collective modes would not be obvious without duality, but is confirmed by expanding the r.h.s. of (8) in powers of the renormalised α (fermionic mass). The dominant free bosonic term is the conductivity of the *massless* Thirring model (i.e. a quantum wire). For a finite length wire with appropriate boundary conditions the IR singularities of this leading term (now exact) are controllable and calculable[9], and the predicted conductance can be observed experimentally. For our *massive* Thirring model, analysis[8] of $D_R^{SG}(k_0,k)$ shows that it, in addition, contains the full bosonic self-energy which, through duality, is related to the fermion condensate $\ll \bar{\psi}\psi \gg$ and the net average fermion density $\rho(T,\mu)$, which we have already calculated. Further results will be presented elsewhere[8].

4 Conclusions

We have tried to summarise some intriguing relations which hold between the $T>0$, $\mu\neq 0$ MT model, and the SG model and Coulomb gases. The particular link with a 1D CG for $T\gtrsim m$ has enabled many thermodynamic quantities to be obtained analytically as discussed in section 2, and we also made some first steps at tackling transport coefficients in section 3 in terms of the dual modes.

The relationship with the CG also provided an interesting interpretation of chiral symmetry restoration in the MT model in terms of binding of ‘chiral charges’ $\hat{\sigma}_{\pm}$. Can any of these pictures be of relevance to QCD, for which the analogue Coulomb gas is one of monopoles[10]? We believe that a tentative answer to that question is yes, particularly for the understanding of chiral symmetry restoration in QCD.

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